Math 241

Problem Set 6 solution manual

Exercise. A6.1

We will describe $\phi : \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$, as a matrix $M = \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$. Then $\phi(x, y, z) = (x, y, z). \begin{bmatrix} a & d \\ b & e \\ c & f \end{bmatrix}$.(all calculations being mod 2)

Example 1: $H =< (1, 0, 0) >= \{(0, 0, 0), (1, 0, 0)\} = \{(x, y, z) \mid y = z = 0\}$
We can take $\phi(x, y, z) = (y, z) = (x, y, z) \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. So $M_H = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example 2: $H =< (1, 0, 1) >= \{(0, 0, 0), (1, 0, 1)\} = \{(x, y, z) \mid y = 0(mod2) \;\; x = z\} = \{(x, y, z) \mid y = 0(mod2) \;\; x + z = 0\}$
We can take $\phi(x, y, z) = (y, x + z) = (x, y, z) \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. So $M_H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Example 3: $H = \{(0, 0, 0), (0, 0, 1), (1, 1, 0), (1, 1, 1)\} = \{(x, y, z) \mid y = x\} = \{(x, y, z) \mid x + y = 0\}$
We can take $\phi(x, y, z) = (x + y, 0) = (x, y, z) \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$. So $M_H = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$.

H	The matrix M_H
$< (1,0,0) >$	$\begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$
$< (1,1,0) >$	$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$
$< (1,1,1) >$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}$
$< (1,0,1) >$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$
$< (0,1,0) >$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 1 \end{bmatrix}$
$< (0,1,1) >$	$\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$
$< (0,0,1) >$	$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \end{bmatrix}$
$\{(0,0,0),(1,0,0),(1,1,0),(0,1,0)\}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$
$\{(0,0,0),(1,0,0),(1,1,1),(0,1,1)\}$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$
$\{(0,0,0),(1,0,0),(1,0,1),(0,0,1)\}$	$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$
$\{(0,0,0),(0,1,0),(1,1,1),(1,0,1)\}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}$
$\{(0,0,0),(0,1,0),(0,1,1),(0,0,1)\}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$
$\{(0,0,0),(0,0,1),(1,1,0),(1,1,1)\}$	$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$
$\{(0,0,0),(1,1,0),(0,1,1),(1,0,1)\}$	$\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$

Section. 11

Exercise. 10

Notice that in $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$, every elements is of order 2.

The subgroups of $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ are:

$<(1,0,0)>=\{(0,0,0),(1,0,0)\}$.
$<(1,1,0)>=\{(0,0,0),(1,1,0)\}$.
$<(1,1,1)>=\{(0,0,0),(1,1,1)\}$.
$<(1,0,1)>=\{(0,0,0),(1,0,1)\}$.
$<(0,1,0)>=\{(0,0,0),(0,1,0)\}$.
$<(0,1,1)>=\{(0,0,0),(0,1,1)\}$.
$<(0,0,1)>=\{(0,0,0),(0,0,1)\}$.

Also we have the subgroups of order 4 :
$<(1,0,0),(0,1,0)>=<(1,0,0),(1,1,0)>=<(0,1,0),((1,1,0)>=\{(0,0,0),(1,0,0),(1,1,0),(0,1,0)\}$.
$\{(0,0,0),(1,0,0),(1,1,1),(0,1,1)\}$.
$\{(0,0,0),(1,0,0),(1,0,1),(0,0,1)\}$.
$\{(0,0,0),(0,1,0),(1,1,1),(1,0,1)\}$.
$\{(0,0,0),(0,1,0),(0,1,1),(0,0,1)\}$.
$\{(0,0,0),(0,0,1),(1,1,0),(1,1,1)\}$.
$\{(0,0,0),(1,1,0),(0,1,1),(1,0,1)\}$.

Exercise. 11

The subgroups of order 4 are:
We have to cyclic subgroups:
$<(0,1)>=\{(0,0),(0,1),(0,2),(0,3)\}$,
$<(1,1)>=\{(0,0),(1,1),(1,2),(0,3)\}$.

And also we have the following subgroup: $\{(0,0),(1,0),(0,2),(1,2)\}$, which is isomorphic to the klein group.

Exercise. 16

We know that : $\mathbb{Z}_{12} \cong \mathbb{Z}_3 \times \mathbb{Z}_4$
Also : $\mathbb{Z}_6 \cong \mathbb{Z}_3 \times \mathbb{Z}_2$
Then : $\mathbb{Z}_2 \times \mathbb{Z}_{12} \cong \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_4$.
And : $\mathbb{Z}_4 \times \mathbb{Z}_6 \cong \mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_2$
Then : $\mathbb{Z}_4 \times \mathbb{Z}_6 \cong \mathbb{Z}_2 \times \mathbb{Z}_{12}$.

Exercise. 17

The maximum order of an element is the order of (1,1,1) (check!), which is of order the $lcm(8,10,24) = 120$.

Exercise. 18

They are not isomorphic since , let $G_1 = \mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$
$G_2 = \mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$.

Notice that :
In $\mathbb{Z}_8$ we have 4 elements whose order is a divisor of 4 (0,2,4, and 6).
In $\mathbb{Z}_{10}$ we have 2 elements whose order is a divisor of 4 (0, and 5).
In $\mathbb{Z}_{24}$ we have 4 elements whose order is a divisor of 4 (0,6,12, and 18).

So in $\mathbb{Z}_8 \times \mathbb{Z}_{10} \times \mathbb{Z}_{24}$ we have 32 elements whose order is a divisor of 4.

In $\mathbb{Z}_4$ we have 4 elements whose order is a divisor of 4 (0,1,2, and 3).
In $\mathbb{Z}_{12}$ we have 4 elements whose order is a divisor of 4 (0,3,6 and 9).
In $\mathbb{Z}_{40}$ we have 4 elements whose order is a divisor of 4 (0,10,20, and 30).

So in $\mathbb{Z}_4 \times \mathbb{Z}_{12} \times \mathbb{Z}_{40}$ we have 64 elements whose order is a divisor of 4.
Then the two groups G_1, and G_2 can't be isomorphic.

Exercise. 46
Let us prove it by induction:

base step: for n=2 :Let $(G_1, \star)$, and $(G_2, *)$ be two abelian groups, and let us prove that $G_1 \times G_2$ is abelian.

let (a, b), and $(c, d) \in G_1 \times G_2$. Since G_1 and G_2 are abelian we have $a \star b = b \star a$, and $c * d = d * c$ then we get $(a, b)(c, d) = (a \star b, c * d) = (b \star a, d * c) = (b, d)(a, c)$.

$\implies G_1 \times G_2$ is abelian.

Inductive step: Suppose it is true up to n, and let us prove it for $n + 1$.

Let $G_1, ..., G_n, G_{n+1}$ be n abelian groups , by the inductive step we have $G_1 \times G_2 \times ... \times G_n$ is abelian , call it G, then we have $G_1 \times G_2 \times ... \times G_n \times G_{n+1} = G \times G_{n+1}$, then by the same reasoning in the base step we get that $G_1 \times G_2 \times ... \times G_{n+1}$ is abelian.

Exercise. 47

Given G abelian, and H is equal to the set of all elements of order 2 together with the identity. Let us prove H is a subgroup of G.

- $e \in H$ by definition of H.
- let $a, b \in H$, (WLOG we can consider both a,and b are both not e) , we have a is of order 2 , and b is of order 2, and $(ab)^2 = (ab)(ab) \underset{associative}{=} a(ba)b \underset{abelian}{=} a(ab)b = (aa)(bb) = e$ then ab is also of order 2, then $ab \in H$.
- Let $a \in H$ then $a^2 = e \implies a.a = e \implies a = a^{-1}$, then $a^{-1} \in H$.

And Hence H is a subgroup of G.

Section. 13

Exercise. 1

ϕ is a homomorphism, since ϕ has the property $\phi(a+b) = a+b = \phi(a)+\phi(b)$.

Exercise. 2

ϕ is not a group homomorphism since $\phi^{-1}(0) = [0,1)$ but $[0,1)$ is not a subgroup of $\mathbb{R}$ under addition.

Exercise. 3

ϕ is a homomorphism since ϕ has the property $\phi(a.b) = |a.b| = |a|.|b| = \phi(a)\phi(b)$.

Exercise. 8

ϕ is not a group homomorphism, since let G be a non-abelian group , and choose $a, b \in G$ such that $a.b \neq b.a$, then $\phi(a.b) = b^{-1}.a^{-1}$, while $\phi(a)\phi(b) = a^{-1}.b^{-1}$, now suppose $b^{-1}.a^{-1} = a^{-1}.b^{-1}$ this implies that $a.b = b.a$ which is not the case. So ϕ is not a homomorphism.

Exercise. 18

We have $\phi : \mathbb{Z} \longrightarrow \mathbb{Z}_{10}$, and $\phi(1) = 6$.

Then since any $x \in \mathbb{Z}$ is either $x = 1 + ... + 1$ or $x = -1... - 1$ (finite sum of x terms), we can see that $\phi(x) = x\phi(1) = 6x$ ($\phi(x) = 6._{10}x$).

This implies that if $x \in ker\phi$ we have $6x = 0 \pmod{10}$, now since 6 has order 5 in $\mathbb{Z}_{10}$, then $ker\phi = \{x \mid 6x = 0 \pmod{10}\}$= $5\mathbb{Z}$.

Finally $\phi(18) = 18._{10}6 = 8$.

Exercise. 22

We have $\phi((1,0)) = 3$, and $\phi((0,1)) = -5$, then $\phi((n,m)) = \phi(n(1,0)+m(0,1)) = n\phi((1,0)) + m\phi((0,1)) = 3n - 5m$.

Then suppose $\phi((x,y)) = 0$, $\implies 3x - 5y = 0$, so $ker\phi = \{(x,y) \in \mathbb{Z}\times\mathbb{Z} \mid 3x - 5y = 0\}$.

$\phi(-3,2) = -3(3) - 5(2) = -19$.

Note that $ker\phi$ is cyclic, generated by (5,3), Reason : $3x + 5 = 0 \implies 3x$ is divisible by 5, $\implies x$ is divisible by 5 $\implies \exists\ l \in \mathbb{Z}$ such that $x = 5l$, but then $5y = 3x = 15l$, or $y = 3l$, so $(x,y) = (5l, 3l) = l(5,3)$.

Exercise. 47

We already know that $ker\phi$ is a subgroup of G, but since $|G|$ is prime then the only subgroups of G are 0, and G ,then $ker\phi$ can either be 0, or G. If $ker\phi = 0$ then ϕ is injective, and if $ker\phi = G$ then $\phi = 0$.

Exercise. 49

We have ϕ: $G \longrightarrow G'$, and ϕ': $G' \longrightarrow G''$ are two homomorphisms.

It is easy to see that $\phi' o\phi$ is a well defined mapping from G into G'', so we only need to check the following:

Let $x, y \in G$ we need to prove that $\phi' o\phi(x.y) = \phi' o\phi(x).\phi' o\phi(y)$.

Start with $\phi' o\phi(x.y) = \phi'(\phi(x.y)) = \phi'(\phi(x).\phi(y)) = \phi'(\phi(x)).\phi'(\phi(y)) = \phi' o\phi(x).\phi' o\phi(y)$

This implies that the composite of two homomorphisms is a homomorphism.

Exercise. 53

ϕ is a homomorphism if and only if h ,and k commute (i,e hk=kh).

-Let ϕ be a homomorphism , then $\phi((1,1)) = \phi((1,0))\phi((0,1))h.k$, but $\phi((1,1)) = \phi((0,1))\phi((1,0)) = kh$. which implies that $hk = kh$.

-Now suppose we have $hk = kh$, then we need to prove ϕ is a homomorphism, then consider $\phi((a,b)+(c,d)) = \phi((a+c, b+d)) = h^{a+c}k^{b+d} = h^a h^c k^b k^d = h^a k^b h^c k^d = \phi((a,b))\phi((c,d))$. Notice that here we used the fact that $h^c.k^b = k^b.h^c$ because h, and k commute, which can be proved by induction on b and c.

Then ϕ is a homomorphism.